Management 3
Quantitative Methods

OPTIMIZATION:
Production, Revenue, Costs, and Profits
The Demand Function

Quantity demanded “Q” or “q” is an inverse function of price “P” or “p”. This is referred to as the “Law of Demand”, i.e. downward-sloping demand.

Price “up” ↑ Quantity demanded “down” ↓ and vice versa.
The Demand Function

For example $Q = 8 - P$. Charting this gives us:

<table>
<thead>
<tr>
<th>Price</th>
<th>Quantity</th>
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<tbody>
<tr>
<td>8</td>
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Revenue

Total Revenue “TR” = \( P \times Q \)

If \( Q = 8 - P \), then

\[ P = 8 - Q \]

And \( P \times Q \)

\[ TR = (8 - Q) \times Q = 8Q - Q^2 \]

This graphs as an inverted parabola as indicated by the negative coefficient on \( Q^2 \)
Revenue

\[ TR = P \times Q = 8Q - Q^2 \]

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<th>Price</th>
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</table>
What happens to Sales when Price declines?

What is the $\Delta TR$ for a $\Delta P$?

<table>
<thead>
<tr>
<th>Price</th>
<th>Quantity</th>
<th>Revenue</th>
<th>The $\Delta TR$</th>
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<tbody>
<tr>
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Revenue increases then decreases.
TR is max at $P=4$ $Q=4$

TR at $P$ = $6$ and $Q$ = $2$
Marginal Revenue is a derivative

$$MR = \frac{dTR}{dQ} = \frac{dPQ}{dQ} = \frac{d(8Q - Q^2)}{dQ}$$

Apply the Power Rule to $8Q - Q^2$

And get $8 - 2Q$

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<tr>
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<th>Revenue</th>
<th>MR</th>
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Optimizing Functions

The procedure for optimizing a function:
1) Write the *objective* function in terms of the *argument variable*, e.g. Quantity produced “Q”
2) Take the first derivative
3) Set the 1st derivative = 0
4) Solve for the *argument variable*, “Q”
Maximizing Sales

1) Write the function: \( TR = P \cdot Q = 8Q - Q^2 \)

2) Take the first derivative: \( dTR = 8 - 2Q \)

3) Set the first derivative equal to zero: \( 8 - 2Q = 0 \)

4) Solve for the argument: \( Q^*_R = 4 \)

What is maximum \( TR \)?

Insert \( Q^*_R \) into the \( TR \) function

\[
TR^* = 8Q^*_R - Q^*_R^2 = 8(4) - 4^2 = 32 - 16 = 16
\]

This matches our charts and we can verify it by calculating \( TR \) for \( Q=3 \) and \( Q=5 \), the points to the left and the right of \( Q^* \)
1) The first derivative $dTR = MR$ (marginal revenue) i.e. the $\Delta$Revenue for a unit $\Delta$Quantity;

1) Revenue is “max” when $dTR = MR$ is zero;

1) As long as marginal revenue is “+” positive, i.e. $\Delta TR/\Delta Q > 0$, we should increase production.
What about Costs?

There are four basic costs and two variations:

1. **Fixed Costs “FC”**. Constant costs regardless of the level of “Q”;

2. **Variable Costs “VC”**. Costs that increase with “Q”;

3. **Average Total Cost “ATC”**. Total Cost / Quantity;

4. **Marginal Cost “MC”**. The change in TC for a change in Q.

The two variations are AFC and AVC.
The Cost Function

The Cost function looks like this:

$$TC = FC + VC(Q)$$

Costs increase with output “Q”.

There are four basic costs and two variations:

1. Costs increase with production, i.e. are a positive function of Q.
2. The TC curve is upward-sloping;
3. The MC and all of the AC curves are typically non-linear with variations in slope, typically downward-sloping initially, then upward-sloping.
Profit?

Profit is the difference between Sales and Costs.

\[ \pi = TR - TC \]

formally

\[ \pi (Q) = P(Q) * Q - FC - VC(Q) \]

\[ \pi = P * Q - FC - VC \]

All we need to do is insert any level of \( Q \) into the operationalized Demand and Cost functions.
For Example

We have Demand as: \( P = 8 - Q \), and if we have Cost \( TC = 2 + 1Q \), then

\[
\pi = TR - TC
\]

\[
= (8 - Q)Q - (2 + 1Q)
\]

\[
= 8Q - Q^2 - 2 - Q
\]

\[
\pi = 7Q - Q^2 - 2
\]

This is the objective function for profit.

What is the profit-maximizing level of output \( Q^* \)?
Maximum Profit?

Apply the optimization procedure.

- Start with the objective function:
  \[ \pi = 7Q - Q^2 - 2 \]

- Take the first derivative:
  \[ d\pi = 7 - 2Q \]

- Set the 1st derivative = 0 and solve for Q:
  \[ d\pi = 0 = 7 - 2Q \], thus
  \[ Q^*_{\pi} = 3.5 \]

  Notice that this is less than \( Q^*_R = 4.0 \)