Management 3
Quantitative Methods

The Time Value of Money
Part 1B
The Basic Relationship

The earning power of a dollar in-hand today is:

\[ FV = PV \times (1+r)^t \]

where there are two $dollar values:

1. **FV** = future value in $dollars
2. **PV** = present value, $ dollars in-hand today

Which are connected by:

- Time “t” and
- A rate-of-interest “r”.
Derive “r” and “t” from the basic equation

\[ r = \left[ \frac{FV}{PV} \right]^{1/t} - 1 \tag{3} \]

and

\[ t = \ln \left( \frac{FV}{PV} \right) / \ln(1+r) \tag{4} \]
\[ r = \left[ \frac{FV}{PV} \right]^{1/t} - 1 \]
FV/ PV

This is Total Return.

Start w/ $100. Invest it.

Get $150 in 3 years = \( \frac{150}{100} = 1.5x \)

We have a multiple of 1.5x our starting value:

$100 of initial investment plus 50% return on Investment (“ROI”).
Return on Investment “ROI”

Total Return minus 1.

\[(FV/ PV) -1 = 1.5 - 1.0 = 0.5 = 50\%\]

This was achieved over 3 years.

How to compare it to other Returns.

Annualize it.

Break it down to its annual ROI.
\[ r = \left[ \frac{FV}{PV} \right]^{1/t} - 1 \]

We need to scale the 3 years to 1 year.

\[ r = 1.5^{1/3} - 1 \]

\[ r = 0.1447 = 14.47\% \text{ per year} \]

Check it ....
\[ r = 1.5^{1/3} - 1 \]
\[ r = 0.1447 = 14.47\% \text{ per year} \]

\[ \$100.00 \times (1+0.1447) = \$114.47 \]
\[ \$114.47 \times (1+0.1447) = \$131.03 \]
\[ \$131.03 \times (1+0.1447) = \$149.99 \]

\[ = \]

\[ \$100.00 \times (1.1447)^3 = \$150.00. \]
\[ r = \left( \frac{FV}{PV} \right)^{1/t} - 1 \]

If we know the two dollar values, Future and Present, and the time that connects them, then this formula gives us the “rate” that we earned if we started with $PV$ and ended with $FV$ after “t” years.
What annual interest rate will double money in: 8 years

use

\[ r = \left[ \frac{FV}{PV} \right]^{1/t} - 1 \]

\[ = 2^{(0.125)} - 1 \]

\[ = 9.05 \text{ percent} \]
\[ r = \left[ \frac{FV}{PV} \right]^{1/t} - 1 \]

Calculate the rate that doubles your money in 12 years.

\[ r = \left[ \frac{2}{1} \right]^{1/12} - 1 \]
\[ r = [2]^{0.0833} - 1 \]
\[ r = \]
What about non-uniform annual Returns?

Invest

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$100</td>
<td>$110</td>
<td>$120</td>
<td>$130</td>
</tr>
<tr>
<td>Annual Return</td>
<td>10.0%</td>
<td>9.1%</td>
<td>8.3%</td>
<td></td>
</tr>
</tbody>
</table>

What is the average annual return year-by-year? It is

\[ 9.13\% = \left[ \frac{130}{100} \right]^{1/3} - 1 = \]

\[ = \left[ (1+r_1) \times (1+r_2) \times (1+r_3) \right]^{1/3} - 1 \]

\[ = \left[ (1.10) \times (1.091) \times (1.083) \right]^{1/3} - 1 \]
What about non-uniform annual Returns?

Invest $100

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$100</td>
<td>$50</td>
<td>$100</td>
</tr>
<tr>
<td>Annual Return</td>
<td>-50%</td>
<td>+100%</td>
<td></td>
</tr>
</tbody>
</table>

What is the average annual return year-by-year?
It is \((-0.50 + 1.0)/2 = 0.50/2 = 0.25 = 25\%\)
But this is not true! The ROI is zero.
What about non-uniform annual Returns?

\[= \left[ (1+r_1)x(1+r_2) \right]^{1/2} - 1\]

\[= \left[ (1-0.50)x(1+1.0) \right]^{1/2} - 1\]

\[= \left[ 0.50x(2.0) \right]^{1/2} - 1\]

\[(1.0)^{1/2} - 1 = 1 - 1 = 0\]
\[ t = \frac{\ln \left( \frac{FV}{PV} \right)}{\ln (1+r)} \]

If we know the $PV$ and the $FV$ and the "rate" applied to the $PV$ each year, the this formula tells us what "\( t \)" in years connected all of this.
Solving for “t”

How long will it take to double your money at an interest rate of: 9 percent

t = ln (FV/PV) / ln(1+r)

t = ln (2/1) / ln(1+0.09)

t = ln (2) / ln (1.09)

= 0.6931 / 0.0861

= 8.04 years
Analyzing Single Amounts

• **Determine FV** given you have a PV, time, and a rate.

• **Determine PV** given a promised FV, time, and a rate.

• **Determine return** “r” for a certain ending FV, starting PV, and duration “t” between the two.

• **Determine time** “t” it will take to earn a return “r”, on a certain ending FV and starting PV.
The four derivations from FV

[1] \[ \text{FV} = \text{PV} \times (1+r)^t \] where \((1+r)^t\) is a FVF

[2] \[ \text{PV} = \text{FV} \times \frac{1}{(1+r)^t} \] where \(\frac{1}{(1+r)^t}\) is a PVF

[3] \[ r = \left[ \frac{\text{FV}}{\text{PV}} \right]^{1/t} - 1 \]

[4] \[ t = \ln \left( \frac{\text{FV}}{\text{PV}} \right) / \ln(1+r) \]
Returns on Investment – what’s the “r”?

• Invest $100 and receive $175 three years from now.

1. What is your return?
2. What is your “annualized” return?
   – Return = (FV/PV) - 1 = (175/100) - 1 = 1.75 - 1 = 75%
   – Annualized Return
     = (FV/PV)^(1/t) - 1
     = (175/100)^(1/3) - 1
     = 1.75 ^(0.33) - 1 = 1.205 - 1 = 20.5%
Check the Calculation

- Starting with $100, check the 20% return:
  - $100.00 \times (1.20) = $120.00 \text{ in one year}
  - $120.00 \times (1.20) = $144.00 \text{ in two years}
  - $144.00 \times (1.20) = $172.80 \text{ in three years}

Confirming the “r” is about 20 percent