Management 3
Quantitative Methods

The Time Value of Money
Part 2A

Future Value of Annuities

Revised 2/15/15
New Scenario: Savings Plans

We often want to accumulate a future balance of money in order to pay for something in the future.

We do this with a saving’s plan. We invest a fixed, predetermined, amount on a periodic basis to build-up a fund. This is called a Sinking Fund.

Businesses do this to:
• … meet future pension obligations;
• … make balloon payments on loans;
Annuities

• An annuity is a “fixed” periodic payment or deposit:
  1. $1,000 per year/month for 36 months.

• These payments can be made at the beginning, or at the end, of the financing period:
  a) Annuities “Due” are payments made at the beginning of the period;
  b) “Ordinary” Annuities are payments made at the end of the period.
Annuities

➢ If you win the Lottery, you receive an *Annuity Due* because you get the first payment now.

➢ If you borrow (take a mortgage), your payments are an *Ordinary Annuity* to the lender because your 1st payment is not due the day you borrow, but one month later.
Annuities

- An annuity is defined according to its parameters. We speak of annuities by:

1. Term of the annuity
2. Amount of the annuity, and
3. The expected annual rate of return on the money invested.

For example: A 5-year, $1,000, 5% annuity means that we save $1,000/year for 5 years, and the funds earn 5% per year.
What is the Future Value of an Annuity?

If you Saved $10,000 per year for 5 years how much would you have in 5 years?

It depends.

• At the very least, you should have $10,000 x 5 = $50,000.

• However, if the money earns >0% interest, you would have > $50,000.

• And, the sooner you started saving, the more you would have.
The future value of $10,000 per year (ordinary annuity style) for 5 years, earning 5% per year:

<table>
<thead>
<tr>
<th>Year Begin</th>
<th>Earn 5%</th>
<th>Total</th>
<th>Deposit</th>
<th>Year End</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ 0</td>
<td></td>
<td>$ 0</td>
<td>$ 10,000</td>
<td>$ 10,000</td>
</tr>
<tr>
<td>$ 10,000</td>
<td>(1.05)</td>
<td>$ 10,500</td>
<td>$ 10,000</td>
<td>$ 20,500</td>
</tr>
<tr>
<td>$ 20,500</td>
<td>(1.05)</td>
<td>$ 21,525</td>
<td>$ 10,000</td>
<td>$ 31,525</td>
</tr>
<tr>
<td>$ 31,525</td>
<td>(1.05)</td>
<td>$ 33,101</td>
<td>$ 10,000</td>
<td>$ 43,101</td>
</tr>
<tr>
<td>$ 43,101</td>
<td>(1.05)</td>
<td>$ 45,256</td>
<td>$ 10,000</td>
<td>$ 55,256</td>
</tr>
</tbody>
</table>
A Shorter Way?

The first $10,000 would earn 4 years @ 5%
This is because it is deposited at the end of year 1.
The second $10,000 would earn 3 years @ 5%
This is because it is deposited at the end of year 2.
The third $10,000 would earn 2 years @ 5%
The fourth $10,000 would earn 1 year @ 5%
The fifth $10,000 would be your last deposit

• This looks like the sum of four calculations using FV Factors from Table 1 x $10,000 each plus the last payment.
The FV Annuity Table is just a sum of FVF’s

- Below are the FVF’s from Table 1 where $r = 5\%$ and $t= 5$ years.
- If we start w/ 1.0 then add the first four from Table 1, we have a new Factor, the FVF of an annuity.

<table>
<thead>
<tr>
<th>Time</th>
<th>5.0%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to 1</td>
<td>1.0000</td>
</tr>
<tr>
<td>1 to 2</td>
<td>1.0500</td>
</tr>
<tr>
<td>2 to 3</td>
<td>1.1025</td>
</tr>
<tr>
<td>3 to 4</td>
<td>1.1576</td>
</tr>
<tr>
<td>4 to 5</td>
<td>1.2155</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>5.5256</strong></td>
</tr>
</tbody>
</table>
The $10,000/year for 5 years @ 5 percent

- We know that a 5 year $10,000 annuity at 5% has a FV of $55,256 because

<table>
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<tr>
<td></td>
<td><strong>5.5256</strong></td>
</tr>
</tbody>
</table>

$5.5256 \times $10,000 = $55,256

$10,000 \times \text{FVFA}(5\%, 5) = $55,256
The FVFA Table

Table 3 is a Table of Future Value Factors for Annuities written FVFA (r, t)
Each Factor, is defined by its rate “r” and its time in years “t”:

\[ FVFA(r, t) = \left( \left[ \frac{FVF(r, t) - 1}{r} \right] \right) \]

\[ FVFA(r, t) = \left[ \frac{[(1+r)^t - 1]}{r} \right] \]

These are called Future Value Factors of Annuities
and are found on the FVFA Table 3
The 5 year @ 5 percent FVFA

\[
FVFA(r, t) = \frac{[FVF(r, t) - 1]}{r}
\]

\[
FVFA(5\%, 5) = \frac{[(1+5\%)^5 - 1]}{5\%}
\]

\[
FVFA(5\%, 5) = \frac{[(1.05)^5 - 1]}{0.05}
\]

\[
FVFA(5\%, 5) = \frac{[1.2763 - 1]}{0.05}
\]

\[
FVFA(5\%, 5) = \frac{[0.2763]}{0.05}
\]

\[
FVFA(5\%, 5) = 5.5256
\]
Formulation: the Future Value of an annuity is

Annuity \times \text{FVFA} (r, t) = \text{Future Value}

[Dollars/Year] \times \left[ \frac{[(1+r)^t - 1]}{r} \right] = \text{Dollars}

We can rearrange this:

Annuity = \frac{\text{Future Value}}{\left[ \frac{[(1+r)^t - 1]}{r} \right]}

Or, not as useful

\text{FVFA} = \left[ \frac{[(1+r)^t - 1]}{r} \right] = \frac{\text{Future Value}}{\text{Annuity}}
Reverse the problem. Say that you want $55,256 in 5 years. Interest rates are 5%. How much would you need to save each year?

Use this formulation:

\[
\text{Annuity} = \frac{\text{Future Value}}{\left[ \frac{1}{1+r} -1 \right]} /r \\
\]

And insert the data:

Annuity = $55,256 / FVFA(5\%, 5)$

Annuity = $55,256 / 5.5256$

$10,000 = \frac{55,256}{5.5256}$
The FV Annuity Table

- The FVFA – future value factor annuity – Table is 1.0 plus the sum of the FVF’s – out first Table - up to any point the “t” years column.

- The FVFA’s will always be a number greater than the number of years, since FVF’s are each $> 1$. 